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Linear Interpolation of the Functions with Three Variable Values with Simple Nodes

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Abstract

The theme of interpolation of a function with three variable values is a complex matter which implies major difficulties when solving it concerning its form , the grade of the polynomial interpolation and the number of points from the definition field .The paper presents two cases of linear interpolation which may be applied when solving problems with elementary fields such as pyramidal.

Key words: *interpolation, network, division, divided differences*

Introduction

Considering a function $f:[a,b]\to \mathbf{R}$ and a division of the interval $[a,b]$

$$\Delta:(x_1=a<x_2<x_3<...<x_n=b)$$

for which it is known the point $A_k(x_k,y_k)$ with $y_k=f(x_k)$ for $k\in\{1,2,...,n\}$.

It is asked to be determined a polynomial of the degree n named P which should approximate the function *f* on the interval $[a,b]$ so as:

$$P(x_k)=y_k=f(x_k) \text{ for } k\in\{1,2,...,n\}. \tag{1}$$

The answer to this problem is given by the formula of Lagrange [3]:

$$P(x)=\sum_{k=1}^{n} y_k \frac{Q_k(x)}{Q_k(x_k)} \tag{2}$$

where

$$Q_k(x)=\frac{\omega(x)}{x-x_k} \text{ and } \omega(x)=\prod_{i=1}^{n}(x-x_i) \tag{3}$$

or by the formula of Newton [3] using the divided differences:

$$\begin{aligned} P(x)=P(f;x_1,x_2,...,x_n;x)=f[x_1]+f[x_1,x_2]\cdot(x-x_1)+f[x_1,x_2,x_3]\cdot \\ \cdot(x-x_1)(x-x_2)+...+f[x_1,x_2,...,x_n]\cdot(x-x_1)(x-x_2)...(x-x_{n-1}) \end{aligned} \tag{4}$$

where the divided differences are:

$$f[x_1]=f(x_1) \; and \quad f[x_1,x_2,...,x_k]=\frac{f[x_1,...,x_{k-1}]-f[x_2,...,x_k]}{x_1-x_k} \tag{5}$$

or

$$f[x_1,x_2,...,x_k]=\int_0^1 dt_1\left(\int_0^{t_1} dt_2\left(...\int_0^{t_{k-2}} f^{(k-1)}(x_1+t_1(x_2-x_1)+...+t_{k-1}(x_k-x_{k-1}))dt_{k-1}...\right)\right) \tag{6}$$

Content

A. Considering the field $D=[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]\subset \mathbf{R}^3$ with the divisions

$$\Delta_1 : x_0=a_1<x_1<x_2<...<x_i<x_{i+1}<...<x_n=b_1;$$

$$\Delta_2 : y_0=a_2<y_1<y_2<...<y_j<y_{j+1}<...<y_m=b_2;$$

$$\Delta_3 : z_0=a_3<z_1<z_2<...<z_k<z_{k+1}<...<z_p=b_3 ,$$

which determine the division:

$$\Delta=\Delta_1\times\Delta_2\times\Delta_3$$

for the field D , and the function $f:D\rightarrow\mathbf{R}$ for which it is known $f(x_i,y_j,z_k)\overset{not.}{=}z_{i,j,k}$, with $i\in\{0,1,...,n\}$, $j\in\{0,1,...,m\}$ and $k\in\{0,1,2,....p\}$.

The notions of the network Δ points notes $N(x_i,y_i,z_k)\overset{not.}{=}N_{i,j,k}$, and

$$D_{i,j,k}\overset{not.}{=}\{(x,y,z)|x_i\le x\le x_{i+1} \;,y_j\le y\le y_{j+1} \;,z_k\le z\le z_{k+1}\},$$

for $i\in\{0,1,...,n\}$, $j\in\{0,1,...,m\}$ and $k\in\{0,1,2,....p\}$.

Being determined a linear polynomial basis $P^l_{i,j,k}$ ($l\in\{1,2,3,4,5,6,\}$) in x , y and z which has to aproximate the function f on the field $D_{i,j,k}$ .The field $D_{i,j,k}$ is divided in six pyramidal subfields in the following manner:

$D^1\overset{not.}{=}D^1_{i,j,k}$ is determines of the points $N_{i+1,j,k+1};N_{i+1,j,k};N_{i,j,k};N_{i+1,j+1,k}$;

$D^2\overset{not.}{=}D^2_{i,j,k}$ is determines of the points $N_{i+1,j,k+1};N_{i,j,k+1};N_{i,j,k};N_{i+1,j+1,k+1}$;

$D^3\overset{not.}{=}D^3_{i,j,k}$ is determines of the points $N_{i+1,j,k+1};N_{i+1,j+1,k};N_{i,j,k};N_{i+1,j+1,k+1}$;

$D^4\overset{not.}{=}D^4_{i,j,k}$ is determines of the points $N_{i+1,j+!,k};N_{i,j+!,k};N_{i,j,k};N_{i,j+1,k+1}$;

$D^5 \overset{not.}{=} D^5_{i,j,k}$ is determines of the points $N_{i,j,k+1}; N_{i,j+1,k+1}; N_{i,j,k}; N_{i+1,j+1,k+1}$;

$D^6 \overset{not.}{=} D^6_{i,j,k}$ is determines of the points $N_{i+1,j+1,k}; N_{i,j+1,k+1}; N_{i,j,k}; N_{i+1,j+1,k+1}$.

Evidently, we have: $D_{i,j,k} = \bigcup_{l=1}^{6} D^l_{i,j,k} = \bigcup_{l=1}^{6} D^l$, and the linear polynomial $P^l_{i,j,k}$ approximate the function f on the field $D^l = D^l_{i,j,k}$, where $P^l_{i,j,k}: D^l \to \mathbf{R}$ for $l \in \{1,2,3,4,5,6,\}$.

The polynomial $P^l_{i,j,k}$ has the form:

$$P^l_{i,j,k}(x,y,z) = \alpha \cdot (x - x_i) + \beta(y - y_j) + \gamma(z - z_k) + \delta \text{ , for } l \in \{1,2,3,4,5,6,\}. \tag{7}$$

For $l = 1$, we imposing the following conditions in the nodes of pyramid D^1.

$$\begin{cases} P^1_{i,j,k}(x_{i+1}, y_j, z_{k+1}) = u_{i+1,j,k+1} \ ; \\ P^1_{i,j,k}(x_{i+1}, y_j, z_k) = u_{i+1,j,k} \ ; \\ P^1_{i,j,k}(x_i, y_j, z_k) = u_{i,j,k} \ ; \\ P^1_{i,j,k}(x_{i+1}, y_{j+1}, z_k) = u_{i+1,j+1,k} \ . \end{cases} \tag{8}$$

It is obtained a system in the unknowns $\alpha, \beta, \gamma, \delta$. Solving the system (8) and using the notations from [4] it is obtained:

$$\begin{cases} \delta = u_{i,j,k} \ ; \\ \alpha = \dfrac{u_{i+1,j,k} - u_{i,j,k}}{x_{i+1} - x_i} = f[x_i, x_{i+1}; y_j; z_k] \ ; \\ \gamma = \dfrac{u_{i+1,j,k+1} - u_{i+1,j,k}}{z_{k+1} - z_k} = f[x_{i+1}; y_j; z_k, z_{k+1}] \ ; \\ \beta = \dfrac{u_{i+1,j+1,k} - u_{i+1,j,k}}{y_{j+1} - y_j} = f[x_{i+1}; y_j, y_{j+1}; z_k] \ . \end{cases} \tag{9}$$

Replacing in (7) it is obtained:

$$P^1_{i,j,k}(x,y,z) = f[x_i, x_{i+1}; y_j; z_k] \cdot (x - x_i) + f[x_{i+1}; y_j, y_{j+1}; z_k] \cdot (y - y_j) +$$
$$+ f[x_{i+1}; y_j; z_k, z_{k+1}] \cdot (z - z_k) + u_{i,j,k} \text{ , for } (x,y,z) \in D^1, \tag{10}$$

and represents interpolation polynomial of the function *f* on the field D^1.

For $l = 2$, we imposing the following conditions in the nodes of pyramid D^2:

$$\begin{cases} P^2_{i,j,k}(x_{i+1}, y_j, z_{k+1}) = u_{i+1,j,k+1} \ ; \\ P^2_{i,j,k}(x_i, y_j, z_{k+1}) = u_{i,j,k+1} \ ; \\ P^2_{i,j,k}(x_i, y_j, z_k) = u_{i,j,k} \ ; \\ P^2_{i,j,k}(x_{i+1}, y_{j+1}, z_{k+1}) = u_{i+1,j+1,k+1} \ . \end{cases} \tag{11}$$

It is obtained a system in the unknowns $\alpha, \beta, \gamma, \delta$. Solving the system (11), using the notations from [4] it and replacing in (7) it is obtained:

$$P^2_{i,j,k}(x,y,z) = f[x_i, x_{i+1}; y_j; z_{k+1}] \cdot (x - x_i) + f[x_{i+1}; y_j, y_{j+1}; z_{k+1}] \cdot (y - y_j) +$$

$$+ f[x_i; y_j; z_k, z_{k+1}] \cdot (z - z_k) + u_{i,j,k}, \text{ for } (x,y,z) \in D^2, \tag{12}$$

and represents interpolation polynomial of the function *f* on the field D^2.

For $l = 3$, we imposing the following conditions in the nodes of pyramid D^3:

$$\begin{cases} P^3_{i,j,k}(x_{i+1}, y_j, z_{k+1}) = u_{i+1,j,k+1} \ ; \\ P^3_{i,j,k}(x_{i+1}, y_{j+1}, z_{k+1}) = u_{i+1,j+1,k} \ ; \\ P^3_{i,j,k}(x_i, y_j, z_k) = u_{i,j,k} \ ; \\ P^3_{i,j,k}(x_{i+1}, y_{j+1}, z_k) = u_{i+1,j+1,k} \ . \end{cases} \tag{13}$$

It is obtained a system in the unknowns $\alpha, \beta, \gamma, \delta$. Solving the system (13), using the notations from [4] and replacing in (7) it is obtained:

$$P^3_{i,j,k}(x,y,z) = (-\frac{(z_{k+1} - z_k)(y_{j+1} - y_j)}{x_{i+1} - x_i} f[x_{i+1}; y_j, y_{j+1}; z_k, z_{k+1}] + f[x_i, x_{i+1}; y_j; z_k]) \cdot$$

$$\cdot (x - x_j) + f[x_{i+1}; y_j, y_{j+1}; z_{k+1}] \cdot (y - y_j) +$$

$$+ f[x_{i+1}; y_{j+1}; z_k, z_{k+1}] \cdot (z - z_k) + u_{i,j,k} \ , \ (x,y,z) \in D^2, \tag{14}$$

and represents interpolation polynomial of the function *f* on the field D^3.

For $l = 4$, we imposing the following conditions in the nodes of pyramid D^4:

$$\begin{cases} P^4_{i,j,k}(x_{i+1}, y_{j+!}, z_k) = u_{i+1,j+1,k} \ ; \\ P^4_{i,j,k}(x_i, y_{j+1}, z_k) = u_{i,j+1,k} \ ; \\ P^4_{i,j,k}(x_i, y_j, z_k) = u_{i,j,k} \ ; \\ P^4_{i,j,k}(x_i, y_{j+1}, z_{k+1}) = u_{i,j+1,k+1} \ . \end{cases} \tag{15}$$

It is obtained a system in the unknowns $\alpha, \beta, \gamma, \delta$. Solving the system (15), using the notations from [4] and replacing in (7) it is obtained:

$$P_{i,j,k}^{4}(x,y,z) = f[x_i, x_{i+1}; y_{j+1}; z_k] \cdot (x - x_i) + f[x_i; y_j, y_{j+1}; z_k] \cdot (y - y_j) +$$

$$+ f[x_i; y_{j+1}; z_k, z_{k+1}] \cdot (z - z_k) + u_{i,j,k}, \text{ for } (x,y,z) \in D^4, \tag{16}$$

and represents interpolation polynomial of the function *f* on the field D^4.

For $l = 5$, we imposing the following conditions in the nodes of pyramid D^5:

$$\begin{cases} P_{i,j,k}^{5}(x_i, y_j, z_{k+1}) = u_{i,j,k+1} \ ; \\ P_{i,j,k}^{5}(x_i, y_{j+1}, z_{k+1}) = u_{i,j+1,k+1} \ ; \\ P_{i,j,k}^{5}(x_i, y_j, z_k) = u_{i,j,k} \ ; \\ P_{i,j,k}^{5}(x_{i+1}, y_{j+1}, z_{k+1}) = u_{i+1,j+1,k+1} \ . \end{cases} \tag{17}$$

It is obtained a system in the unknowns $\alpha, \beta, \gamma, \delta$. Solving the system (17), using the notations from [4] and replacing in (7) it is obtained:

$$P_{i,j,k}^{5}(x,y,z) = f[x_i, x_{i+1}; y_{j+1}; z_{k+1}] \cdot (x - x_i) + f[x_i; y_j, y_{j+1}; z_{k+1}] \cdot (y - y_j) +$$

$$+ f[_i; y_j; z_k, z_{k+1}] \cdot (z - z_k) + u_{i,j,k} \quad \text{, for } (x,y,z) \in D^5 \tag{18}$$

and represents interpolation polynomial of the function *f* on the field D^5.

For $l = 6$, we imposing the following conditions in the nodes of pyramid D^6:

$$\begin{cases} P_{i,j,k}^{6}(x_{i+1}, y_{j+1}, z_{k+1}) = u_{i+1,j+1,k+1} \ ; \\ P_{i,j,k}^{6}(x_{i+1}, y_{j+1}, z_k) = u_{i+1,j+1,k} \ ; \\ P_{i,j,k}^{6}(x_i, y_j, z_k) = u_{i,j,k} \ ; \\ P_{i,j,k}^{6}(x_i, y_{j+1}, z_{k+1}) = u_{i,j+1,k+1} \ . \end{cases} \tag{19}$$

It is obtained a system in the unknowns $\alpha, \beta, \gamma, \delta$. Solving the system (19), using the notations from [4] and replacing in (7) it is obtained:

$$P_{i,j,k}^{6}(x,y,z) = f[x_i, x_{i+1}; y_{j+1}; z_{k+1}] \cdot (x - x_i) +$$

$$+(-\frac{(z_{k+1} - z_k)(x_{i+1} - x_i)}{y_{j+1} - y_j} f[x_i; x_{i+1}; y_{j+1}; z_k, z_{k+1}] + f[x_i; y_j, y_{j+1}; z_k]) \cdot (y_{j+1} - y_j) +$$

$$+ f[x_{i+1}; y_j; z_k, z_{k+1}] \cdot (z - z_k) + u_{i,j,k}, \text{ for } (x,y,z) \in D^6 \tag{20}$$

and represents interpolation polynomial of the function *f* on the field D^6.

The polynomial basis $P_{i,j,k}^{l}$ ($l \in \{1,2,3,4,5,6,\}$) interpolates the function f on the field $D_{i,j,k} = \bigcup_{l=1}^{6} D_{i,j,k}^{l} = \bigcup_{l=1}^{6} D^{l}$, with $i \in \{0,1,...,n-1\}$, $j \in \{0,1,...,m-1\}$ and $k \in \{0,1,2,....p\}$.

$$P_{i,j,k}(x,y,z) = \begin{cases} P_{i,j,k}^{1}(x,y,z) & for \ \ (x,y,z) \in D^{1} \quad (10) \\ P_{i,j,k}^{2}(x,y,z) & for \ \ (x,y,z) \in D^{2} \quad (12) \\ P_{i,j,k}^{3}(x,y,z) & for \ \ (x,y,z) \in D^{3} \quad (14) \\ P_{i,j,k}^{4}(x,y,z) & for \ \ (x,y,z) \in D^{4} \quad (16) \\ P_{i,j,k}^{5}(x,y,z) & for \ \ (x,y,z) \in D^{5} \quad (18) \\ P_{i,j,k}^{6}(x,y,z) & for \ \ (x,y,z) \in D^{6} \quad (20) \end{cases} \tag{21}$$

with $i \in \{0,1,...,n-1\}$, $j \in \{0,1,...,m-1\}$ and $k \in \{0,1,2,....p\}$.

In the particular case when the divisions Δ_1, Δ_2 and Δ_3 are equidistant, namely:

$$x_{i+1} - x_i = \frac{b_1 - a_1}{n} = h_1 \ , \ y_{j+1} - y_j = \frac{b_2 - a_2}{m} = h_2 \ , \ z_{k+1} - z_k = \frac{b_3 - a_3}{p} = h_3 \ ,$$

with $i \in \{0,1,...,n-1\}$, $j \in \{0,1,...,m-1\}$ and $k \in \{0,1,2,....p\}$ and using the divided differences, then , the error made when the function *f* replaced by the polynomial $P_{i,j,k}$ given by (21) is:

$$e_{\tau} = \left| f(x,y,z) - P_{i,j,k}(x,y,z) \right| \le 2 \cdot M_1 \cdot \left(h_1^{2} + h_2^{2} + h_3^{2} + h_1 h_2 + h_1 h_2 + h_2 h_3 \right) \tag{22}$$

where:

$$M_1 = \max_{(x,y,z)\in D} \{ \left| \frac{\partial^2 f}{\partial x^2}(x,y,z) \right| ; \left| \frac{\partial^2 f}{\partial y^2}(x,y,z) \right| ; \left| \frac{\partial^2 f}{\partial z^2}(x,y,z) \right| ; \left| \frac{\partial^2 f}{\partial x \partial y}(x,y,z) \right| ; \left| \frac{\partial^2 f}{\partial x \partial z}(x,y,z) \right| ; \left| \frac{\partial^2 f}{\partial y \partial z}(x,y,z) \right| \} .$$

B. Taking intro account the case in which *D* is an elementary field, there is $a_1 \le b_1$; $a_2 \le b_2$; $a_3 \le b_3$, so $D \subseteq D' = [a_1,b_1] \times [a_2,b_2] \times [a_3,b_3] \subset \mathbf{R}^3$. For the field D' we consider the division $\Delta = \Delta_1 \times \Delta_2 \times \Delta_3$ from the case A .The nodes of the division Δ may be:

inner nodes if $N_{i,j,k} \in D$;

outer nodes if $N_{i,j,k} \in D' - D$, for $i \in \{0,1,...,n\}$, $j \in \{0,1,...,m\}$ and $k \in \{0,1,2,....p\}$.

We continue the function $f : D \to \mathbf{R}$ on the field D' this way $\overline{f} : D' \to \mathbf{R}$ with :

$$\overline{f}(N_{i,j,k}) = \begin{cases} 0 & if \;\; N_{i,j,k} \;\; is \;\; an \;\; outer \;\; node \;\; , \\ f(N_{i,j,k}) & if \;\; N_{i,j,k} \;\; is \;\; an \;\; inner \;\; node \;\; . \end{cases}$$

For the function $\overline{f}$ we apply the theoretical results from the previous case.

Conclusions

The paper presents two cases of interpolation of a function with three variable values, with grade one polynomial and determines the linear interpolation on pyramidal subfields this being the most frequently used method of interpolation.

References

1. B u c u r, M. - *Metode numerice*, Editura Facla, Timişoara, 1973
2. D e m i d o v i c i, B.P. - *Computational Mathematics*, Editura Mir, Moscova, 1970
3. D i n u, T. - *Analiză numerică*, Editura U.P.G. din Ploişti, 2002
4. D i n u, T. – Interpolation of the functions with two variable values with simple nodes*, Buletinul Universității Petrol-Gaze din Ploieşti, Seria Matematică-Informatică-Fizică*, nr.1/2007, pp. 7-12, 2007
5. D i n u, T. - Linear and quadratic interpolation of the functions with two variable values with simple, *Buletinul Universității Petrol-Gaze din Ploieşti, Seria Matematică-Informatică-Fizică*, nr. 2/2007, pp. 17-22, 2007
6. G r i g o r e, Gh. - *Lecții de Analiză numerică*, Facultatea de Matematică din Bucureşti (curs litografiat), 1984
7. M a r i n e s c u, Gh. – *Analiză numerică*, Editura Didactică şi Pedagogică, Bucureşti, 1974
8. M a r i n e s c u, Gh. – *Probleme de analiză numerică,* Editura Didactică şi Pedagogică, Bucureşti, 1974

Interpolarea liniară a funcțiilor cu trei variabile cu noduri simple

Rezumat

Problema interpolării unei funcții de trei variabile este o problemă care prezintă dificultăți majore în rezolvare în funcție de formă, gradul polinomului de interpolare şi de numărul punctelor din domeniul de definiție. În lucrare sunt prezentate două cazuri de interpolare liniară cu aplicabilitate mare în probleme cu domenii elementare de tip piramidă.